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Dielectric tensor and magnetic permeability in the weak field approximation of general relativity†

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Abstract. We present a treatment of classical electromagnetic theory in the presence of a weak gravitational field, both in a vacuum and in a material medium, in terms of an effective dielectric and magnetic permeability tensor. We show that the gravitational red shift can be interpreted as the work done by the electric field of the light ray against the gravitationally induced polarisation current. We derive the dispersion relation for an electromagnetic wave in a medium and show that it depends upon the polarisation state of light.

1. Introduction

In a series of papers (Pegoraro *et al* 1978a, Pegoraro *et al* 1978b, Iacopini *et al* 1979)‡ we have indicated how the effects due to the interaction between the gravitational and the electromagnetic fields could be used to detect gravitational waves. Electromagnetic detectors, as an alternative to mechanical and to interferometrical ones, were first suggested by Braginskii (Braginskii and Menskii 1971a, b, Braginskii *et al* 1974) and their approach was subsequently revived by several authors (Kulak 1978, Caves 1979) and discussed theoretically by Tourrenc (1975a, b).

The detectors proposed in papers I and II exploit the energy transfer between two levels of an electromagnetic resonator induced by the gravitational wave when its frequency equals the frequency difference between the levels. In paper III we have instead suggested the possibility of detecting the birefringence induced in a medium by the strains caused by the gravitational wave. In the discussion of both detectors we have found it convenient to use the concept of a dielectric tensor associated, in a vacuum or in matter, with a gravitational wave. In this way we were able to exploit the analogy with electromagnetic phenomena in anisotropic media and with the phenomena which arise in electronic devices with time-dependent elements. In this paper we wish to present a more detailed derivation of this approach and apply it to the interpretation of some optical phenomena in curved space.

It is well known (Landau and Lifshitz 1962, Møller 1969) that in the presence of a gravitational field the propagation of an electromagnetic wave in a vacuum is modified in a way which is similar to that caused by a material medium. For example, a ray of light is bent by an external gravitational field, in the same way as in a medium with a space-dependent refraction index. It seems natural therefore to describe the electromagnetic field in the presence of gravitation, both in a vacuum and in a material

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‡ These papers will be referred to in the following as papers I, II and III respectively.

medium, by means of four (three-dimensional) vector fields, ***E***, ***B***, ***D*** and ***H***, linearly related by a permeability tensor, in the same way as one does in a medium at rest in the presence of gravity. In the latter case the field ***D*** is a linear function depending on ***E*** alone, of the form†

$$D^i = \epsilon_{\rm M}^{ik} E_k = -(\epsilon_{\rm M})^i_j \eta^{jk} E_k \tag{1.1}$$

where $\epsilon_{\rm M}$, which in general depends on the space and time coordinates, is the dielectric tensor of the matter, and η^{ij} is the Euclidean metric. A similar relation holds between ***B*** and ***H***.

We shall show in § 2 that in a vacuum, in the presence of gravity, one can define two fields, ***D*** and ***H***, depending linearly on the electric and magnetic fields through a permeability tensor which is a function of the metric field. In particular, for a metric with the time coordinate orthogonal to the three-dimensional space manifold, i.e. such that $g_{i0} = 0$, we shall show that the relation between ***E*** and ***D*** is of the form (I, II)

$$D^i = \epsilon_{\rm G}^{ik} E_k = f(g_{\mu\nu}) g^{ik} E_k \tag{1.2}$$

where f is a function of the metric tensor $g_{\mu\nu}$ which reduces to -1 in Minkowski space, i.e. when $g_{\mu\nu} = \eta_{\mu\nu}$. In this case the geometrical dielectric tensor $\epsilon_{\rm G}$ reduces, as expected, to the identity.

It is natural to think of equations (1.1) and (1.2) as the limiting cases of a general relation valid when both matter and gravity‡ are present. The dielectric tensor relating ***D*** to ***E*** will, in the general case, describe the effects of both matter and gravity and will reduce to (1.1) in matter in the Minkowski limit, and to (1.2) in a vacuum in the presence of gravity. The total dielectric tensor can then be written as

$$\epsilon^{ik} = (\tilde{\epsilon}_{\rm M})^i_j \epsilon_{\rm G}^{jk} = f(g_{\mu\nu})(\tilde{\epsilon}_{\rm M})^i_j g^{jk} \tag{1.3}$$

where $\tilde{\epsilon}_{\rm M}$ differs from the gravity-free tensor $\epsilon_{\rm M}$ because of the strains caused in the medium by the gravitational field. A similar relation holds for the magnetic permeability.

So far we have not mentioned the boundary conditions which the electromagnetic field must satisfy. Suppose the boundary conditions are time dependent: due to the covariance of Maxwell's equations under general coordinate transformations, it is possible to transform them to a reference system where the boundary conditions are time independent. This corresponds to changing the metric tensor and therefore to changing the dielectric constant.

When one considers the effect of the gravitational field on an electromagnetic field confined by material walls, one has to consider not only the effect described in equation (1.2) but also the effect of the motion of the walls caused by the gravitational field. Neither the former effect nor that of the walls is independent of the coordinate system. It can be shown that the dielectric constant resulting from the combination of the two effects is defined modulo the transformations that leave the walls invariant. Throughout this paper we shall assume that the reference frame has been chosen in such a way that the walls and the matter contained within them will be at rest.

In § 2 we shall rewrite the generally covariant Maxwell equations in a vacuum in the presence of a gravitational field in a way which is formally equal to the equation for the

† We shall use the following conventions: Greek indices run from 0 to 3, Latin indices from 1 to 3; repeated indices are summed; the signature in Minkowski space is $(1, -1, -1, -1)$; the velocity of light c and Planck's constant $\hbar$ are set equal to one.

‡ The gravitational field is assumed to be weak so we can treat it to first order only.

electromagnetic field in the presence of matter in flat space. In the case of a weak gravitational field we will derive the expressions for the 'polarisation' charge and current and show that, in terms of them, the Lagrangian of the electromagnetic field in a vacuum takes the familiar form of the Lagrangian in flat space for an electromagnetic field interacting with a field-dependent current.

In § 3 we shall apply the formalism developed in the previous section to show that the red shift caused by a static gravitational field can be interpreted as arising from the work done by the electric field of a light packet against the polarisation current induced in a vacuum by the gravitational field. Although the gravitational potential is static, in the frame where the time coordinate coincides with the proper time of the source and of the detector, the refraction index and the amplitude of the polarisation current are time dependent, thus giving rise to an energy transfer.

The dielectric properties of a medium under the influence of the gravitational field are discussed in § 4. Besides the effects, already analysed in § 2, which are due to the change of geometry, one must also take into account in this case the modification of the dielectric tensor caused by the strains induced in the medium by the gravitational field.

In § 5 we derive the dispersion relation satisfied by an electromagnetic wave propagating in a medium under the influence of a gravitational field. We shall show that a material medium becomes birefringent and this effect, as has been shown in paper III, can be applied to the detection of gravitational waves.

2. The gravitational permeability tensor of the vacuum

Let us consider the electromagnetic field in the presence of an external gravitational field. The generally invariant action of the electromagnetic field in a vacuum is (Landau and Lifshitz 1962)

$$A=\int \mathscr{L}(x)\,\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 \tag{2.1}$$

where $\mathscr{L}(x)$ is the Lagrangian density at the point $x=(x^0=t, x^1, x^2, x^3)$:

$$\mathscr{L}(x)=(16\pi)^{-1}\sqrt{-g}\,F_{\mu\nu}(x)F^{\mu\nu}(x). \tag{2.2}$$

Here $g=\det g_{\mu\nu}$ is the determinant of the metric tensor, and $F_{\mu\nu}(x)$ is the covariant electromagnetic field tensor related to the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ in the usual way:

$$E_i(x)=-F_{0i}(x), \qquad B^i(x)=-\tfrac{1}{2}\epsilon^{ijk}F_{jk}(x), \tag{2.3}$$

where $-\epsilon^{ijk}=\epsilon_{ijk}$ is the completely antisymmetrical Ricci symbol.

To solve Maxwell's equations obtained from the Lagrangian (2.2), one needs to specify the boundary conditions satisfied by the fields $\boldsymbol{E}$ and $\boldsymbol{B}$. In general these conditions are time dependent: we shall assume that the coordinate system has been chosen in such a way as to eliminate the time dependence of the boundary conditions.

By analogy with the method (Weisskopf 1936) which is used to define the material vectors $\boldsymbol{D}$ and $\boldsymbol{H}$ in a medium, we introduce the tensor density $H_{\mathrm{G}}^{\mu\nu}$,

$$H_{\mathrm{G}}^{\mu\nu}=8\pi\,\partial\mathscr{L}/\partial F_{\mu\nu}=\sqrt{-g}\,F^{\mu\nu}(x), \tag{2.4}$$

and the two vector fields

$$D^i(x)=H_{\mathrm{G}}^{0i}(x), \qquad H_i(x)=\tfrac{1}{2}\epsilon_{ijk}H_{\mathrm{G}}^{jk}(x). \tag{2.5}$$

One easily verifies that D^i and H_i satisfy the first set of Maxwell's equations, as do their counterparts defined for a material medium. Equation (2.4) can be written as a linear relation between $H_G^{\mu\nu}$ and $F_{\mu\nu}$:

$$H_G^{\mu\nu} = \tfrac{1}{2}\sqrt{-g}(g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta})F_{\alpha\beta} = \kappa_G^{\mu\nu\alpha\beta}(g_{\rho\sigma})F_{\alpha\beta}. \tag{2.6}$$

The Lagrangian (2.2) then reads

$$\mathscr{L} = \frac{1}{16\pi}F_{\mu\nu}H_G^{\mu\nu} = \frac{1}{16\pi}F_{\mu\nu}\kappa_G^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{2.7}$$

and is therefore formally equal to the Lagrangian of the electromagnetic field in the absence of gravity in a medium with a geometrical permeability tensor κ_G which depends upon the metric tensor $g_{\mu\nu}$.

We shall only discuss the effects due to the gravitational permeability of the vacuum in the weak field approximation, i.e. to first order in the deviation $h_{\mu\nu}$ of the metric tensor from the Minkowski tensor $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{2.8}$$

where, consistent with the weak field approximation, $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$.

To first order in $h_{\mu\nu}$ the permeability tensor reads:

$$\kappa_G^{\mu\nu\alpha\beta} = \tfrac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\nu\alpha}\eta^{\mu\beta}) - \chi_G^{\mu\nu\alpha\beta} \tag{2.9}$$

where the susceptibility χ_G is a linear function of $h_{\rho\sigma}$:

$$\chi_G^{\mu\nu\alpha\beta} = \tfrac{1}{2}[h^{\mu\alpha}\eta^{\nu\beta} + h^{\nu\beta}\eta^{\mu\alpha} - h^{\nu\alpha}\eta^{\mu\beta} - h^{\mu\beta}\eta^{\nu\alpha} - \tfrac{1}{2}h_{\rho\sigma}\eta^{\rho\sigma}(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\nu\alpha}\eta^{\mu\beta})]. \tag{2.10}$$

Using (2.9) and (2.10) the Lagrangian (2.7) takes the form

$$\mathscr{L} = \frac{1}{16\pi}F_{\mu\nu}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta} - \frac{1}{16\pi}F_{\mu\nu}\chi_G^{\mu\nu\alpha\beta}F_{\alpha\beta} = \mathscr{L}_0 - \tfrac{1}{2}h^{\mu\nu}T_{\mu\nu}. \tag{2.11}$$

The first term, $\mathscr{L}_0$, represents the electromagnetic Lagrangian in flat space, whereas the second is the 'minimal' interaction between the metric field $h^{\mu\nu}$ and the electromagnetic energy–momentum tensor $T_{\mu\nu}$ evaluated to zero order in $h_{\mu\nu}$:

$$T_{\mu\nu} = (4\pi)^{-1}(F_{\mu\alpha}\eta^{\alpha\beta}F_{\nu\beta} - \tfrac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}\eta^{\alpha\rho}\eta^{\beta\sigma}F_{\rho\sigma}). \tag{2.12}$$

The interaction term in equation (2.11) can be put in a different form by defining the polarisation current induced in a vacuum by the gravitational field:

$$J_G^{\mu} = \frac{1}{4\pi}\frac{\partial}{\partial x^{\nu}}(\eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta} - H_G^{\mu\nu}) = \frac{1}{4\pi}\frac{\partial}{\partial x^{\nu}}(\chi_G^{\mu\nu\alpha\beta}F_{\alpha\beta}). \tag{2.13}$$

In terms of J_G^{μ}, the Lagrangian (2.11) can be written in the more familiar form

$$\mathscr{L} = \mathscr{L}_0 - \tfrac{1}{2}J_G^{\mu}A_{\mu} \tag{2.14}$$

where A_μ is the vector potential. The Lagrangian (2.14) is obtained from (2.11) by integrating the interaction term by parts and by dropping a four-dimensional divergence. The factor $\frac{1}{2}$ in the interaction term arises from the linear dependence of J_G^{μ} on the vector potential A_μ: indeed when $\mathscr{L}$ is varied with respect to A_μ, the current cannot be kept constant.

In terms of the vectors $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{E}$ and $\boldsymbol{B}$, equation (2.6) reads:

$$D^i = \epsilon_G^{ij}E_j + \lambda^i{}_j B^j, \qquad H_i = (\mu_G^{-1})_{ij}B^j - E_j\lambda^j{}_i, \tag{2.15}$$

where

$$\epsilon_{\mathrm{G}}^{ij} = -2\kappa_{\mathrm{G}}^{0i0j} = -\sqrt{-g}(g^{00}g^{ij} - g^{i0}g^{0j}),$$
$$(\mu_{\mathrm{G}}^{-1})_{ij} = \tfrac{1}{2}\epsilon_{ilm}\kappa_{\mathrm{G}}^{lmpq}\epsilon_{pqj} = \tfrac{1}{2}\sqrt{-g}\,\epsilon_{ilm}g^{lp}g^{mq}\epsilon_{pqj}, \tag{2.16}$$
$$\lambda^{i}{}_{j} = \kappa_{\mathrm{G}}^{0ilm}\epsilon_{lmj} = g^{ik}g^{0l}\epsilon_{lkj}.$$

In three-dimensional language the polarisation charge and current densities are

$$4\pi\rho_{\mathrm{G}} = -\frac{\partial}{\partial x^{i}}(\eta^{ij}E_j + \epsilon_{\mathrm{G}}^{ij}E_j + \lambda^{i}{}_{j}B^{j}),$$

$$4\pi J_{\mathrm{G}}^{i} = \epsilon^{ijk}\frac{\partial}{\partial x^{j}}[(\eta_{kl} + (\mu_{\mathrm{G}}^{-1})_{kl})B^{l} - E_l\lambda^{l}{}_{k}] + \frac{\partial}{\partial t}[(\eta^{ij} + \epsilon_{\mathrm{G}}^{ij})E_j + \lambda^{i}{}_{j}B^{j}]. \tag{2.17}$$

Equations (2.15) and (2.17) are reminiscent of the relations which occur in the case of a moving material medium. In particular, the terms proportional to λ are the equivalent of those arising from the drag velocity.

The above relations take a simpler form in the case of a time-orthogonal system (Møller 1969), i.e. when $g_{i0} = 0$, which is the equivalent of a medium at rest. In these systems $\lambda^{i}{}_{j} = 0$, $\boldsymbol{D}$ depends only on $\boldsymbol{E}$ and $\boldsymbol{H}$ only on $\boldsymbol{B}$:

$$D^{i} = -(-g)^{1/2}g^{00}g^{ij}E_j = \epsilon_{\mathrm{G}}^{ij}E_j, \qquad H_i = -(-g)^{-1/2}g_{00}g_{ij}B^{j} = (\mu_{\mathrm{G}}^{-1})_{ij}B^{j}. \tag{2.18}$$

To obtain these relations, we have written $g = g_{00}\det g_{rs}$ and used the identity

$$\epsilon_{ijk}g^{jl}g^{km}\epsilon_{lmn} = (2/\det g_{rs})g_{in} \tag{2.19}$$

which holds when $g^{i0} = 0$.

Equations (2.18) show that the two fields $\boldsymbol{D}$ and $\boldsymbol{H}$ defined by (2.4) and (2.5) are related to $\boldsymbol{E}$ and $\boldsymbol{B}$ in the same way as in a material medium at rest. However, the 'medium' has the special property that

$$\epsilon_{\mathrm{G}}^{ij} = \mu_{\mathrm{G}}^{ij} = -\sqrt{-g}\,g^{00}g^{ij} \simeq -\eta^{ij} + \tfrac{1}{2}(\mathrm{Tr}\,h_{\mu\nu})\eta^{ij} + h^{ij}. \tag{2.20}$$

The formalism developed in this section has been applied in papers I and II to the case where the metric field $g_{\mu\nu}$ is due to a monochromatic gravitational wave with amplitude h and frequency Ω. In this case the dielectric tensor ϵ_{G} is a sinusoidal function of time. As a consequence, the capacity (and the inductance) of an electromagnetic cavity varies with time as $C = C_0(1 + h\cos\Omega t)$. When the cavity forms part of a two-level resonator with an appropriate geometry, the time dependence of the capacity establishes a parametric coupling between the two levels and energy is thus pumped from one level to the other.

The energy transfer between the two levels can also be understood in terms of the polarisation charge and current. Consider an electromagnetic cavity with two levels with frequencies ω, $\omega + \Omega$ and let us suppose that, initially, only the lower frequency is excited. The second equation (2.17) shows that under the effect of a gravitational field oscillating at frequency Ω, a polarisation current is produced containing the frequency $\omega + \Omega$. This current excites the field of the upper level in the cavity.

3. The Poynting theorem and the gravitational red shift

In paper I we have generalised Poynting's theorem to the case of an electromagnetic field in the presence of a weak gravitational field described by a time orthogonal metric tensor. In this section we shall show that this theorem provides a natural interpretation of the red shift. It will appear that the reddening of the photons propagating in a time-independent potential ϕ can be attributed to the work done by the electric field of the wave against the polarisation current introduced in the previous section.

Maxwell's equations in a vacuum in a gravitational field are†

$$\nabla \wedge \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t, \qquad \nabla \wedge \boldsymbol{H} = \partial \boldsymbol{D}/\partial t, \tag{3.1}$$

where the fields $\boldsymbol{D}$ and $\boldsymbol{H}$ are given in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$ by equations (2.15). From equations (3.1) one derives Poynting's theorem in the usual way:

$$\frac{\partial}{\partial t}\frac{\boldsymbol{E}.\boldsymbol{D}+\boldsymbol{B}.\boldsymbol{H}}{8\pi}+\nabla.\frac{\boldsymbol{E}\wedge\boldsymbol{H}}{4\pi}=\frac{-1}{8\pi}(\boldsymbol{E}\dot{\epsilon}_{\mathrm{G}}\boldsymbol{E}-\boldsymbol{B}\dot{\mu}_{\mathrm{G}}^{-1}\boldsymbol{B})-\frac{1}{4\pi}\boldsymbol{E}\dot{\lambda}\boldsymbol{B} \tag{3.2}$$

where the dot denotes the derivative with respect to time. Equation (3.2) differs from equation (2.13) of paper I by the term proportional to $\dot{\lambda}$, which was absent there because of the use of a time-orthogonal metric.

We now want to use equation (3.2) to determine the change in the proper frequency of a photon propagating in a weak, static gravitational potential ϕ, which we shall take to depend only upon $x_3 = z$. The photon proper frequency is measured in a frame where the emitting and the absorbing atoms are at rest, the potential is time independent, and the time coordinate coincides with the atoms' proper time. These conditions are satisfied by a metric tensor of the form

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & u(z,t) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ u(z,t) & 0 & 0 & -1 \end{bmatrix} \tag{3.3}$$

where

$$u(z,t) = t\frac{\mathrm{d}}{\mathrm{d}z}(1+\phi(z)). \tag{3.4}$$

With this choice for the metric, the permeability tensors (2.16) become

$$\epsilon^{ij} = -\eta^{ij} = \mu^{ij}, \qquad \lambda^{i}{}_{j} = \eta^{ik} g^{0l} \epsilon_{lkj} \tag{3.5}$$

so that the terms in brackets in the right-hand side of (3.2) vanish. We remark that although the gravitational potential ϕ is time independent, the metric tensor (3.3), and therefore the drag velocity λ, depend explicitly upon t.

Before deriving the expression for the red shift, we want to give an intuitive interpretation to the remaining source term in equation (3.2) by re-expressing it in terms of the Poynting vector and the gravitational acceleration. By using (2.16) we obtain, to first order in ϕ,

$$-\frac{1}{4\pi}\boldsymbol{E}\dot{\lambda}\boldsymbol{B} \simeq \frac{1}{4\pi}E_i\dot{\lambda}^{i}{}_{j}\eta^{jk}H_k = \frac{1}{4\pi}\epsilon^{mik}E_iH_k\dot{g}^{0l}\eta_{lm} = S^m a_m \tag{3.6}$$

where

$$S^m = \epsilon^{mik}E_iH_k/4\pi \tag{3.7}$$

† The tensor notation for $\nabla \wedge E$ is $(\nabla \wedge \boldsymbol{E})^i = -\epsilon^{ijk}(\partial/\partial x^j)E_k$.

is the Poynting vector and

$$a_m = -\partial\phi/\partial x^m \tag{3.8}$$

is the gravitational acceleration. The term $-(1/4\pi)\boldsymbol{E}\dot{\lambda}\boldsymbol{B}$ therefore represents the rate of work per unit volume done by the electromagnetic field against the gravitational field.

Alternatively the above result can be expressed in terms of the polarisation current. In our frame of reference the polarisation current (2.17) depends only upon λ (see equation (3.5)). The rate of work done by the electric field $\boldsymbol{E}$ against the polarisation current is thus

$$-\int d^3x\, E_i J^i_G = \frac{1}{4\pi}\int d^3x\, E_i\left(\epsilon^{ijk}\frac{\partial}{\partial x^j}(E_l\lambda^l{}_k) - \frac{\partial}{\partial t}(\lambda^i{}_j B^j)\right). \tag{3.9}$$

Using Maxwell's equations, it is easy to prove that equation (3.9) can be rewritten as

$$-\int d^3x\, E_i J^i_G = \frac{-1}{4\pi}\int d^3x\, E_i\dot{\lambda}^i{}_j B^j = \int d^3x\, S^i a_i. \tag{3.10}$$

To derive the gravitational red shift, we consider a wave packet emitted at time $t = 0$, at $z = z_0$ with frequency $\omega(z_0)$. Integrating equation (3.2) over space and using (3.6), we obtain

$$dW/dt = \int d^3x\, \boldsymbol{S}\,.\,\boldsymbol{a} \simeq \boldsymbol{P}\,.\,\boldsymbol{a} \tag{3.11}$$

where W and $\boldsymbol{P}$ are the wave packet's energy and momentum. In the geometrical optics approximation equation (3.11) implies a relation between the wave packet central frequency ω and the progation vector $\boldsymbol{k}$ (Landau and Lifshitz 1962, p 148) of the form

$$d\omega/dt = \boldsymbol{k}\,.\,\boldsymbol{a}. \tag{3.12}$$

Integrating equation (3.12) over the wave packet's path from the point of emission z_0 to that of absorption z_1, we obtain, with the help of (3.8),

$$\frac{\omega(z_1) - \omega(z_0)}{\omega(z_0)} = \phi(z_0) - \phi(z_1) \tag{3.13}$$

which is the well known expression for the red shift. The change of frequency can be traced back to the time dependence of the metric tensor in the frame where t at the positions of the emitting and absorbing atoms coincides with their proper time.

4. The electromagnetic field in a material medium

We want now to generalise the discussion of § 2 to the case where a medium is present. In this case an external gravitational field, besides changing the space geometry as in the vacuum case, modifies the dielectric and magnetic properties of the medium as a result of the elastic strains caused by the gravitational field itself.

In the absence of gravity, and in Minkowski space, a medium at rest is described by the dielectric and magnetic permeability tensors ϵ_M and μ_M which relate the vectors $\boldsymbol{D}$ to $\boldsymbol{E}$ and $\boldsymbol{B}$ and $\boldsymbol{H}$:

$$D^i = -(\epsilon_M)^i_j\eta^{jk}E_k, \qquad H_i = -(\mu_M^{-1})^j_i\eta_{jk}B^k. \tag{4.1}$$

Equations (4.1) have been written in a somewhat pedantic form to preserve the distinction between covariant and contravariant vectors which is essential for a generalisation to curved space.

As a first step towards this goal we write (4.1) in a form valid in any uniformly moving system:

$$H_{\mathrm{M}}^{\mu\nu} = \kappa_{\mathrm{M}}^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{4.2}$$

where $H_{\mathrm{M}}^{\mu\nu}$ is an antisymmetric tensor with components

$$D^i = H_{\mathrm{M}}^{0i}, \qquad H_i = \tfrac{1}{2}\epsilon_{ijk} H_{\mathrm{M}}^{jk}, \tag{4.3}$$

and κ_{M} is the matter permeability tensor in the absence of gravity. Equation (4.2) reduces to (4.1) if, in a medium at rest, the components of the material permeability tensor are given by

$$\begin{aligned} \tfrac{1}{2}\kappa_{\mathrm{M}}^{0i0j} &= (\epsilon_{\mathrm{M}})^i_k \eta^{kj} = (\epsilon_{\mathrm{M}})^i_k \eta^{kj} \eta^{00}, \\ \tfrac{1}{2}\kappa_{\mathrm{M}}^{ijkl} &= \epsilon^{ijr} (\mu_{\mathrm{M}}^{-1})_r^s \epsilon_{spq} \eta^{pk} \eta^{ql}, \end{aligned} \tag{4.4}$$

with

$$\kappa_{\mathrm{M}}^{0ijk} = 0 = \kappa_{\mathrm{M}}^{jk0i}. \tag{4.5}$$

As expected, equations (4.2) and (2.6) coincide in the special case $\epsilon_{\mathrm{M}} = \mu_{\mathrm{M}} = I$, $g_{\mu\nu} = \eta_{\mu\nu}$, i.e. for Minkowski space in a vacuum. It is then natural to consider (4.2) and (2.6) as special cases of a general relationship, valid for a medium in the presence of gravity, connecting the tensor $H^{\mu\nu}$ to the electromagnetic tensor $F_{\alpha\beta}$:

$$H^{\mu\nu} = \kappa^{\mu\nu\alpha\beta} F_{\alpha\beta}. \tag{4.6}$$

Here κ, the total permeability tensor, takes into account both matter and gravity and is obtained from κ_{M} by the following prescriptions:

(i) multiply by $\sqrt{-g}$ to account for the change of the volume element;

(ii) replace the Minkowski tensor $\eta_{\mu\nu}$ by the metric tensor $g_{\mu\nu}$;

(iii) modify the dielectric and magnetic properties of the medium to account for the strains generated in the medium by the gravitational field.

In particular, if, in the system where the medium is at rest, the metric is of the form $g_{i0} = 0$, equation (4.1) gives

$$\begin{aligned} D^i &= (\tilde{\epsilon}_{\mathrm{M}})^i_j \epsilon_{\mathrm{G}}^{jk} E_k = -\sqrt{-g}\, g^{00} (\tilde{\epsilon}_{\mathrm{M}})^i_j g^{jk} E_k, \\ B^i &= \mu_{\mathrm{G}}^{ij} (\tilde{\mu}_{\mathrm{M}})^k_j H_k = -\sqrt{-g}\, g^{00} g^{ij} (\tilde{\mu}_{\mathrm{M}})^k_j H_k, \end{aligned} \tag{4.7}$$

where $\tilde{\epsilon}_{\mathrm{M}}$ and $\tilde{\mu}_{\mathrm{M}}$ are the dielectric and magnetic permeability tensors in the presence of gravity. These tensors differ from their unperturbed (i.e. without gravity) values by terms which, in the linear approximation, are proportional to the strains caused in the medium by gravity. In particular for an isotropic medium, with $(\epsilon_{\mathrm{M}})^i_j = \epsilon\delta^i_j$, $(\mu_{\mathrm{M}})^i_j = \mu\delta^i_j$ (ϵ and μ being two scalars), we can write (Born and Wolf 1959, Landau and Lifshitz 1960):

$$\begin{aligned} (\tilde{\epsilon}_{\mathrm{M}})^i_j &= \epsilon\delta^i_j + a_1[U^i_j - \tfrac{1}{3}(\mathrm{Tr}\, U)\delta^i_j] + a_2(\mathrm{Tr}\, U)\delta^i_j, \\ (\tilde{\mu}_{\mathrm{M}})^i_j &= \mu\delta^i_j + b_1[U^i_j - \tfrac{1}{3}(\mathrm{Tr}\, U)\delta^i_j] + b_2(\mathrm{Tr}\, U)\delta^i_j. \end{aligned} \tag{4.8}$$

In equation (4.8) ϵ, μ and the elasto-optical constants a_1, a_2, b_1, b_2, vary with the frequency ω of the electromagnetic field. The elasto-optical constants describe the electric and magnetic responses of the medium to the mechanical strain U_{ij}. The latter represents instead the elastic response to the gravitational field and depends upon the acoustic frequencies of the medium. Equation (4.8) is valid provided that the frequency ω of the electromagnetic field is much larger than the acoustic frequencies.

The strain tensor U_{ij} is defined by

$$U_{ij} = \tfrac{1}{2}(\partial l_i/\partial x^j + \partial l_j/\partial x^i), \tag{4.9}$$

where

$$l_i(x) = \eta_{ij}\xi^j(x) + \tfrac{1}{2}h_{ij}x^j \tag{4.10}$$

is the effective displacement field which takes into account both the coordinate displacement ξ_i and the change of the metric.

The strain tensor U satisfies the equation (see equation (A.10) of paper I)

$$(\partial^2/\partial t^2 - c_s^2\nabla^2)U_{ij} = -R_{0i0j} \tag{4.11}$$

where R is the Riemann tensor and c_s is the sound velocity of the medium.

In paper III we have considered the solutions of equation (4.11) in the case where the Riemann tensor is due to a monochromatic gravitational wave of frequency Ω. In this case,

$$R_{0i0j} = \tfrac{1}{2}\Omega^2 h_{ij}^{\mathrm{TT}} = \tfrac{1}{2}\Omega^2 A_{ij}\,\mathrm{e}^{-\mathrm{i}\Omega t}, \tag{4.12}$$

where $h_{\mu\nu}^{\mathrm{TT}}$ is the correction to the metric tensor in the transverse traceless gauge (Misner *et al* 1972). To solve equation (4.11) we expand U_{ij} into its eigenmodes (which depend upon the shape of the body),

$$U_{ij} = \sum_n \lambda_n(t)U_{ij}^{(n)}(x), \tag{4.13}$$

and obtain for the amplitude of the nth eigenmode

$$\lambda_n = -\tfrac{1}{2}\,\mathrm{e}^{-\mathrm{i}\Omega t}\frac{\Omega^2}{\Omega^2-\omega_n^2}\frac{1}{V}\int \mathrm{d}^3x\, A_{ij}U_{ij}^{(n)}, \tag{4.14}$$

where ω_n is the frequency of the nth eigenmode and V is the volume of the body. The amplitude λ_n is proportional to the amplitude of the gravitational wave. Since $\mathrm{Tr}\, A_{ij} = 0$, it follows from (4.14) that only quadrupole oscillations can be excited.

5. Polarisation effects

It has been suggested in paper III that the birefringence gravitationally induced in a medium can be used to detect gravitational waves by measuring the difference between the phase velocities of two orthogonally polarised light rays which propagate parallel to a gravitational wave. In this section we derive the dispersion relation on which the method of paper III is based. We show that only in a material medium can the anisotropy induced by a gravitational wave give rise to a birefringence proportional to the amplitude of the gravitational wave. We shall derive the dispersion relation for the case of a slowly varying gravitational field. Equations (3) and (4) of paper III follow as a special case when the gravitational field is that of a gravitational wave.

Let us consider a light beam of frequency ω and wavevector $\boldsymbol{k}$ propagating in a medium.

We shall consider the geometrical optics limit, that is we shall assume $\partial\epsilon/\partial t \ll \omega$ and $\partial\epsilon/\partial x \ll k_i$ (with similar relations for μ).

Let us consider a medium at rest in a time-orthogonal reference frame. From Maxwell's equations and from the relations (4.7) one obtains

$$-\omega^2 D^i = \epsilon^{ijl} k_l \mu^{-1}_{jm} \epsilon^{mnp} k_n \epsilon^{-1}_{pq} D^q, \qquad k_i D^i = 0. \tag{5.1}$$

If we take k_i along the third axis, $k_i = (0, 0, k)$, equation (5.1) depends only on the components of ϵ^{-1} and μ^{-1} in the (1, 2)-plane. We can therefore rewrite it in the form

$$\omega^2 \boldsymbol{D} = k^2 (P\mu^{-1})_{\mathrm{R}} (P\epsilon^{-1}) \boldsymbol{D} \tag{5.2}$$

where P is the projector on the (1, 2)-plane and the subscript R denotes a $\pi/2$ rotation around the third axis. This rotation, which is represented by ϵ^{ij3}, takes into account the fact that the magnetic field $\boldsymbol{H}$ is orthogonal to the field $\boldsymbol{D}$.

We shall first consider the case when the electromagnetic wave propagates in a vacuum, i.e. in a medium with $\epsilon = \mu = \epsilon_{\mathrm{G}}$ (since $\tilde{\epsilon} = \tilde{\mu} = I$, see equation (4.7)). Using the identity

$$A_{\mathrm{R}} A = (\det A) I, \tag{5.3}$$

which is valid for any 2×2 symmetrical matrix, equation (5.2) becomes

$$\omega^2 \boldsymbol{D} = k^2 \det(P\epsilon_{\mathrm{G}}^{-1}) \boldsymbol{D}. \tag{5.4}$$

This shows that in a vacuum, due to the equality $\epsilon = \mu$, the dispersion relation is independent of the polarisation state and therefore no birefringence is produced. Using the explicit expression of ϵ_{G} (see (2.18)), we obtain

$$\det(P\epsilon_{\mathrm{G}}^{-1}) = g^{33}/g^{00}. \tag{5.5}$$

One therefore recovers the dispersion relation for a photon in a gravitational field, namely $k_\mu k^\mu = 0$.

Let us now consider the effect of a material medium where in general $\epsilon \neq \mu$. We shall assume for simplicity that the propagation vector k_i of the electromagnetic wave coincides either with a principal axis of g_{ik}, or with one of $\tilde{\epsilon}$ and $\tilde{\mu}$. If g_{ik} is the field of an unpolarised gravitational wave, the first possibility occurs when the light ray propagates parallel to the gravitational wave. Under this assumption equation (5.2) can be rewritten as

$$\begin{aligned}\omega^2 \boldsymbol{D} &= k^2 (P\tilde{\mu}^{-1})_{\mathrm{R}} (P\tilde{\epsilon}^{-1}) \boldsymbol{D} = k^2 (P\tilde{\mu}^{-1})_{\mathrm{R}} (P\mu_{\mathrm{G}}^{-1})_{\mathrm{R}} (P\epsilon_{\mathrm{G}}^{-1}) (P\tilde{\epsilon}^{-1}) \boldsymbol{D} \\ &= k[\det(P\epsilon_{\mathrm{G}}^{-1})] \,.\, (P\tilde{\mu}^{-1})_{\mathrm{R}} (P\tilde{\epsilon}^{-1}) \boldsymbol{D}.\end{aligned} \tag{5.6}$$

To derive this equation we have used the equality $\epsilon_{\mathrm{G}} = \mu_{\mathrm{G}}$ and the identity (5.3). We notice that because of our assumption about the principal axes of g_{ik} or $\tilde{\epsilon}$ and $\tilde{\mu}$, only the (1, 2)-components of $\tilde{\epsilon}$ and $\tilde{\mu}$ enter equation (5.6). In the special case of a gravitational wave, $\det(P\epsilon_{\mathrm{G}}^{-1}) = 1$ and equation (5.6) is equivalent to equation (4) of paper III.

Equation (5.6) shows that in general the dispersion relation depends upon the light polarisation state. Thus a medium, which in the absence of a gravitational field is isotropic, becomes in general birefringent. This birefringence is due to the strains induced by the gravitational field in the medium.

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